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Four-Dimensional N=2(4) Superstring Backgrounds

and

The Real Heavens

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ABSTRACT

We study N=2(4) superstring backgrounds which are four-dimensional non-Kählerian with non-trivial dilaton and torsion fields. In particular we consider the case that the backgrounds possess at least one $U(1)$ isometry and are characterized by the continual Toda equation and the Laplace equation. We obtain a string background associated with a non-trivial solution of the continual Toda equation, which is mapped, under the T-duality transformation, to the hyper-Kähler Taub-NUT instanton background. It is shown that the integrable property of the non-Kählerian spaces have the direct origin in the real heavens: real, self-dual, euclidean, Einstein spaces. The Laplace equation and the continual Toda equation imposed on quasi-Kähler geometry for consistent string propagation are related to the self-duality conditions of the real heavens with "translational" and "rotational" Killing symmetry respectively.

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1. Introduction

Supersymmetric σ-models have attracted attention for various reasons for a long time. One of them is their deep relationship with complex manifold theory. Recently the interest have been refreshed in connection with superstring theory. It has been shown that the number of supersymmetries realized on two dimensional world sheet restricts the background geometry. If two dimensional world sheet theory has $N = 1$ supersymmetry, no-restriction on the background is imposed. $N = 2$ supersymmetry, however, imposes a number of conditions. The simplest case is that of a torsionless Riemannian background, which must be a Kähler manifold in order to admit $N = 2$ supersymmetry[1]. Such σ-models are conventionally formulated in terms of $N = 2$ chiral superfields and the superspace Lagrangian is just the Kähler potential.

In the presence of torsion, the situation becomes considerably complicated. In this case the background has to admit two covariantly constant complex structures. A typical example of such WZNW σ-models are those with group manifolds as target spaces. In ref.[2] the conditions for $N = 2$ supersymmetry on group manifolds were found and a complete classification was given.

In ref.[4] it was shown that (2,2) supersymmetric σ-models formulated in terms of chiral and twisted chiral superfields describe torsionful target spaces. Moreover, the abelian T-duality transformation was formulated by means of a Legendre transformation which interchanges a chiral superfield with a twisted chiral one in the manifestly $N = 2$ supersymmetry preserving manner. Then the backgrounds which are dual to those described by the familiar (2,2)σ-models formulated in terms of chiral superfields are completely described by ones formulated in terms of chiral and twisted chiral superfields.

In order that these geometries provide consistent string backgrounds they have to satisfy, adding the dilation field, the string equations of motion, namely, the vanishing of β-functions. In ref.[5] a systematic discussion on four-dimensional backgrounds with $N = 2$ world sheet supersymmetry was given. There a set of

conditions were derived, which are imposed on Kähler or torsionful non-Kähler with $N = 2$ world sheet supersymmetry. These conditions for consistent string propagation could be re-expressed by simple differential equations. For example a class of non-Kählerian backgrounds including the axionic instanton background was constructed as solutions to a simple integrable model i.e. one with the Laplace equation as field equation. Following this line, in the presence of (at least) one $U(1)$ isometry, the new four-dimensional non-Kählerian background which has the non-trivial dilaton and torsion fields was constructed in ref.[6]. In this case the constraint imposed on target space geometry is related to an integrable model namely one with the continual Toda equation as field equation and the relation of the solution with the hyper-Kähler Eguchi-Hanson instanton background was discussed.

In this paper, we explore these lines. The new superstring background with non-trivial dilaton and torsion fields, which is dual to the hyper-Kähler Taub-NUT instanton background, is presented. The origin of the integrable property of non-Kählerian backgrounds, which emerge as the Laplace equation and the continual Toda equation, is clarified. It is found that these integrable equations are related to those of the real heavens, which is the self-dual condition of the Riemann curvature of the euclidean Einstein gravity.

This paper is organized as follows.

We begin with a review of some of the relevant aspects presented in ref.[3,4] for constructing non-trivial four-dimensional non-Kählerian backgrounds with torsion fields described by the (2,2) σ-models formulated in terms of one chiral and one twisted chiral superfield. As is worked out in ref.[5], adding dilaton field we present the differential equations imposed on target space geometry for consistent string propagation. Following ref.[6] , it is shown that, due to (at least) one $U(1)$ Killing symmetry, the condition imposed on non-Kählerian backgrounds implies the continual Toda equation. A non-trivial background, which is dual to the Eguchi-Hanson instanton background, is obtained as a solution of the continual

Toda equation. In section 3, it is found that the non-Kählerian background which is dual to the Taub-NUT instanton background can be constructed through a solution of the continual Toda equation. Section 4 is spent to show that the origin of integrability lies in the real heavens. The last section is devoted to a summary and discussions. In the appendix A, the vanishing conditions of β-functions are re-expressed in terms of the quasi-Kähler potential and dilaton field. The duality transformation by means of a Legendre transformation is explained in appendix B.

2. The Quasi-Kähler Geometry and Integrable Equations

2.1. N=2 Superstring backgrounds

The most general $N = 2$ superspace action for one chiral superfield U and one twisted chiral superfield V in two dimensions is determined by a single real function $K(U, \bar{U}, V, \bar{V})$ [3,4]:

$$S = \frac{1}{2\pi\alpha'} \int \mathrm{d}^2 x D_+ D_- \bar{D}_+ \bar{D}_- K(U, \bar{U}, V, \bar{V}). \tag{2.1}$$

The superfields U and V obey a chiral or twisted chiral constraint

$$\bar{D}_\pm U = 0, \qquad \bar{D}_+ V = D_- V = 0. \tag{2.2}$$

The action (2.1) is invariant, up to total derivatives, under the quasi-Kähler gauge transformations:

$$K \to K + \Lambda_1(U, V) + \Lambda_2(U, \bar{V}) + \bar{\Lambda}_1(\bar{U}, \bar{V}) + \bar{\Lambda}_2(\bar{U}, V). \tag{2.3}$$

To read off the target space geometry of the theory it is convenient to write down, denoting u and v as the lowest component of the superfield U and V respectively,

the purely bosonic part of the superspace action (2.1) :

$$S = -\frac{1}{2\pi\alpha'}\int \mathrm{d}^2x[K_{u\bar{u}}\partial^a u\partial_a\bar{u} - K_{v\bar{v}}\partial^a v\partial_a\bar{v} + \epsilon_{ab}(K_{u\bar{v}}\partial_a u\partial_b\bar{v} + K_{v\bar{u}}\partial_a v\partial_b\bar{u})], \quad (2.4)$$

where

$$K_{u\bar{u}} = \frac{\partial^2 K}{\partial U\partial\bar{U}}, \qquad K_{v\bar{v}} = \frac{\partial^2 K}{\partial V\partial\bar{V}}, \qquad K_{u\bar{v}} = \frac{\partial^2 K}{\partial U\partial\bar{V}}, \qquad K_{v\bar{u}} = \frac{\partial^2 K}{\partial V\partial\bar{U}}.$$

The target space metric and anti-symmetric tensor are expressed in terms of K respectively

$$G_{\mu\nu} = \begin{pmatrix} 0 & K_{u\bar{u}} & 0 & 0 \\ K_{u\bar{u}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -K_{v\bar{v}} \\ 0 & 0 & -K_{v\bar{v}} & 0 \end{pmatrix}, \qquad B_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & K_{u\bar{v}} \\ 0 & 0 & K_{v\bar{u}} & 0 \\ 0 & -K_{v\bar{u}} & 0 & 0 \\ -K_{u\bar{v}} & 0 & 0 & 0 \end{pmatrix}.$$

It follows that the field strength $H_{\mu\nu\lambda} = \nabla_\mu B_{\nu\lambda} + \nabla_\nu B_{\lambda\mu} + \nabla_\lambda B_{\mu\nu}$ can also be expressed entirely in terms of the function K:

$$H_{u\bar{u}v} = -\frac{\partial^3 K}{\partial U\partial\bar{U}\partial V}, \; H_{u\bar{u}\bar{v}} = +\frac{\partial^3 K}{\partial U\partial\bar{U}\partial\bar{V}}, \; H_{v\bar{v}u} = +\frac{\partial^3 K}{\partial V\partial\bar{V}\partial U}, \; H_{v\bar{v}\bar{u}} = -\frac{\partial^3 K}{\partial V\partial\bar{V}\partial\bar{U}}.$$

If $K_{u\bar{u}}$ and $K_{v\bar{v}}$ are positive definite, the target space possesses $(2,2)$ signature. To obtain a space with euclidean signature, we have to require that they are positive definite and negative definite respectively. Note that the metric is non-Kählerian with torsion, whereas $N = 2$ world sheet supersymmetry is guaranteed.

Hitherto we have just discussed the geometrical structure of the $N = 2$ supersymmetric σ-models. In string theory, there is another background field, namely the dilaton field $\Phi(u, \bar{u}, v, \bar{v})$, so that one adds to the σ-model action eqn. (2.4) a term of the form $\frac{1}{2}R^{(2)}\Phi(u, v)$, where $R^{(2)}$ is the scalar curvature of the two-dimensional world sheet. In order that these backgrounds provide consistent string solutions, they have to satisfy the vanishing of the β-function equations. Then the

requirement of one-loop conformal invariance of the two-dimensional σ-model leads to the following equations of motion for the background fields [8],

$$
\begin{aligned}
0 = \beta^G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{4} H^{\lambda\sigma}_{\mu} H_{\nu\lambda\sigma} + 2\nabla_\mu \nabla_\nu \Phi + O(\alpha'), \\
0 = \beta^B_{\mu\nu} &= \nabla_\lambda H^\lambda_{\mu\nu} - 2(\nabla_\lambda \Phi) H^\lambda_{\mu\nu} + O(\alpha').
\end{aligned} \tag{2.5}
$$

Moreover, the vanishing of the dilaton β-function is provided by the equation of motion for dilaton field as

$$
0 = \delta c \equiv c - \frac{3D}{2} = \frac{3}{2}\alpha' [4(\nabla\Phi)^2 - 4\nabla^2\Phi - R + \frac{1}{12}H^2] + O(\alpha'^2). \tag{2.6}
$$

In the presence of N=4 world sheet superconformal symmetry, the solution to the lowest order in α' is exact to all orders and δc remains zero to all orders.

The conditions of the vanishing of β-functions were re-expressed in terms of K and Φ entirely in [5]. There three exclusive cases were considered, corresponding to the differential equation which is satisfied by K. In this paper we concentrate ourselves to two of them, the case(i) and case(ii) explained in appendix A. For the case(i), the potential K must satisfy the Laplace equation

$$
K_{u\bar{u}} + K_{v\bar{v}} = 0, \tag{2.7}
$$

and the dilaton field is expressed in terms of a solution K as

$$
2\Phi = \ln K_{u\bar{u}} + \text{const.} \tag{2.8}
$$

In turn the case(ii) is characterized by the following nonlinear differential equation

$$
K_{u\bar{u}} + K_{ww} e^{K_w} = 0, \tag{2.9}
$$

which determines target space geometry. The dilaton field is given in terms of a solution of eqn.(2.9) to be

$$
2\Phi = \ln K_{ww} + \text{const.} \tag{2.10}
$$

In this case quasi-Kähler potential and dilaton field have $U(1)$ Killing symmetry

with respect to W, namely $K = K(u, \bar{u}, w + \bar{w}), \Phi = \Phi(u, \bar{u}, w + \bar{w})$. We denote the $U(1)$ isometry as $U(1)_w$ for simplicity.

2.2. The Hyper-Kähler Eguchi-Hanson Instanton and Integrable Equations

In ref.[6] it was shown that non-Kählerian backgrounds characterized by eqn.(2.9) arise as a solution of the continual Toda equation. Performing the duality transformation its relation to the Eguchi-Hanson instanton background was discussed.

In fact eqn.(2.9) is re-expressed, denoting K_w as $\mathcal{U}$, as the continual Toda equation

$$\partial_u \partial_{\bar{u}} \mathcal{U} + \partial_w^2 e^{\mathcal{U}} = 0. \tag{2.11}$$

Assuming that $\mathcal{U} = \ln \alpha(u, \bar{u}) + \ln \beta(w + \bar{w})$, eqn.(2.11) reduces to the Liouville equation

$$\partial_u \partial_{\bar{u}} \ln \alpha(u, \bar{u}) + k\alpha(u, \bar{u}) = 0 \tag{2.12}$$

where $\partial_w^2 \beta$ is a constant due to the separation of variables and is denoted as k. By using the solution of the Liouville equation authors of ref.[6] employed the simplest non-trivial solution of eqn.(2.11) as

$$\mathcal{U} = \ln \frac{-\rho^2 + (w + \bar{w})^2}{(1 + u\bar{u})^2} \tag{2.13}$$

and wrote down the solution of eqn.(2.9) as

$$\begin{aligned} K = -2(w+\bar{w}) + 2\rho \mathrm{arctanh} \frac{(w + \bar{w})}{\rho} \\ + (w + \bar{w}) \ln(-\rho^2 + (w + \bar{w})^2) - 2(w + \bar{w}) \ln(1 + u\bar{u}). \end{aligned} \tag{2.14}$$

It was shown that after performing the change $w \to -w$ the geometry characterized by (2.14) is dualized with respect to $U(1)_w$ isometry to give the hyper-Kähler Eguchi-Hanson instanton background. However w-sign flipped K is no longer a

solution of eqn.(2.9) . In order to make evident the relation of integrable models to the hyper-Kähler Eguchi-Hanson instanton background, we present here another solution of eqn.(2.9) , which is also associated with the solution (2.13) , as

$$
\begin{aligned}
K = -2(w+\bar{w}) - 2\rho \operatorname{arccoth}\frac{-(w+\bar{w})}{\rho} \\
+ (w+\bar{w})\ln(-\rho^2 + (w+\bar{w})^2) - 2(w+\bar{w})\ln(1+u\bar{u}),
\end{aligned} \tag{2.15}
$$

which describes the non-trivial background with torsion;

$$
\begin{aligned}
ds^2 &= \frac{-4(w+\bar{w})}{-\rho^2+(w+\bar{w})^2}dwd\bar{w} - \frac{4(w+\bar{w})}{(1+u\bar{u})^2}dud\bar{u}, \\
2\Phi &= \ln|\frac{w+\bar{w}}{-\rho^2+(w+\bar{w})^2}| + \text{const.} \\
H_{u\bar{u}w} &= -H_{u\bar{u}\bar{w}} = \frac{+2}{(1+u\bar{u})^2}.
\end{aligned} \tag{2.16}
$$

The scalar curvature is computed to be given by

$$
R = \frac{3\rho^4 - 14\rho^2(w+\bar{w})^2 + 3(w+\bar{w})^4}{2(\rho^2-(w+\bar{w})^2)(w+\bar{w})^3}
$$

which depends only on $w+\bar{w}$ and is asymptotically zero as $w+\bar{w} \to \pm\infty$.

Let us comment on the difference of eqn.(2.15) from eqn.(2.14) . There appears $-\operatorname{arctanh}\frac{(w+\bar{w})}{\rho}$ instead of $\operatorname{arccoth}\frac{-(w+\bar{w})}{\rho}$. Since the geometrical objects are expressed in terms of the derivatives of K, both potentials describe the same geometry except for the range of $(w+\bar{w})^2$, which correspond to describing two different coordinate patches. For the solution (2.14) and (2.15) the range must be $(w+\bar{w})^2 < \rho^2$ and $(w+\bar{w})^2 > \rho^2$ respectively.

There exist different backgrounds with those discussed so far, which are called dual background and obtained by duality transformation. Now we dualize the solution (2.15) with respect to $U(1)$ isometry with respect to the twisted chiral superfield W following the procedure described in the appendix B. The duality transformation interchanges a twisted chiral superfield W with a chiral superfield

Ψ so that the dual theory is described by two chiral superfield. Then dual geometry is Kähler . The dual Kähler potential is determined by

$$\tilde{K} = K - (w + \bar{w})(\psi + \bar{\psi})$$

with a constraint equation $0 = K_w - (\psi + \bar{\psi})$ which determines $(w + \bar{w})$ in terms of $(\psi + \bar{\psi})$ and $u, \bar{u}$.⋆ Now the independent variables are ψ and u.

The constraint is computed to be

$$(w + \bar{w})^2 = \rho^2 + e^{\psi+\bar{\psi}}(1 + u\bar{u})^2 \tag{2.17}$$

which implies $(w + \bar{w})^2 > \rho^2$. This is compatible only when the potential is of the form (2.15). The constraint (2.17) implies that $w + \bar{w} = \pm\sqrt{\rho^2 + e^{\psi+\bar{\psi}}(1 + u\bar{u})^2}$. We use here $w + \bar{w} < 0$ as a solution of (2.17) in order that the geometry (2.16) has euclidean signature.

The dual Kähler potential is computed to be

$$\tilde{K} = 2g - 2\rho \mathrm{arccoth}\frac{g}{\rho}, \qquad g = \sqrt{\rho^2 + e^{\psi+\bar{\psi}}(1 + u\bar{u})^2}. \tag{2.18}$$

Under the coordinate transformation

$$z^1 = e^{\psi/2}, \quad z^2 = e^{\psi/2}u$$

the dual Kähler potential (2.18) describes hyper-Kähler Eguchi-Hanson metric [9] ;

$$g_{1\bar{1}} = 4(\frac{g}{Q^2}|z^2|^2 + \frac{1}{g}|z^1|^2), \qquad g_{1\bar{2}} = -4\frac{\rho^2}{gQ^2}z^2\bar{z}^1, \qquad g_{2\bar{2}} = 4(\frac{g}{Q^2}|z^1|^2 + \frac{1}{g}|z^2|^2), \tag{2.19}$$

where $Q = |z^1|^2 + |z^2|^2$ and $g = \sqrt{\rho^2 + Q^2}$. In other words the hyper-Kähler Eguchi-Hanson instanton background is dual with respect to $U(1)_\psi$ isometry, namely overall $U(1)$ isometry, to the quasi-Kähler background (2.16) .

⋆ We use small letters $w,\ z,\ \psi,$ ... for the lowest component of superfields $W,\ Z,\ \Psi,$ ...

3. The Hyper-Kähler Taub-NUT Instanton and Integrable Equations

We consider in this section the relation of the hyper-Kähler Taub-NUT instanton to the integrable model of the quasi-Kähler geometry. It is shown that the hyper-Kähler Taub-NUT instanton background is also dual to a quasi-Kähler geometry characterized by a solution of the continual Toda equation.

Let us first consider a solution of the continual Toda equation (2.11) with $(u, \bar{u})$ denoted as $(z, \bar{z})$ to avoid confusion in the following discussion;

$$\mathcal{U} = 2\ln[-4(w + \bar{w})\cosh^{-1} 4(z + \bar{z})]. \tag{3.1}$$

The quasi-Kähler potential is

$$K = -2(w + \bar{w}) + 2(w + \bar{w})\ln[-4(w + \bar{w})\cosh^{-1} 4(z + \bar{z})]. \tag{3.2}$$

The corresponding geometry is the same as (2.15) with $\rho = 0$. In fact the quasi-Kähler potential (2.15) with $\rho = 0$ can be transformed to (3.2) under the quasi-Kähler transformation as follows. The potential (2.15) with $\rho = 0$ is expressed as

$$K = -2(w + \bar{w}) + 2(w + \bar{w})\ln[-(w + \bar{w})] - 2(w + \bar{w})\ln[1 + u\bar{u}].$$

We perform the coordinate transformation $u = e^{8z}$ but the resulting potential is not a solution of (2.9) . In order to obtain a solution the quasi-Kähler transformation (A.15) of the form $K \to K + 8(w + \bar{w})(z + \bar{z})$ and (A.16) of the form $K \to K + (w+\bar{w})2\ln 8$ must be followed. Under these transformations we obtain eqn.(3.2) as a solution of (2.9) .

Next let us dualize (3.2) with respect to $U(1)_w$ isometry. We obtain the dual Kähler potential, interchanging a twisted chiral superfield W with a chiral super-

field Z^1 , as

$$\tilde{K} = \frac{1}{2} e^{\frac{1}{2}(z^1+\bar{z}^1)} \cosh \frac{1}{2}(z^2 + \bar{z}^2) \tag{3.3}$$

where we introduce $z^2 \equiv 8z$. The corresponding geometry is completely flat but the coordinate system of eqn.(3.3) suggests the relation to the hyper-Kähler Taub-NUT metric as is seen below.

In order to see the relation of the Kähler potential (3.3) to the hyper-Kähler Taub-NUT metric, let us employ the following expression for the hyper-Kähler Taub-NUT potential [9]

$$K^{TN} = \frac{s}{2}[1 + \frac{1}{4}\lambda s(1 + \cos^2\theta)] \tag{3.4}$$

with the Kähler coordinate

$$\begin{aligned} z^1 &= -i\phi + \ln(s \sin\theta) \\ z^2 &= i\psi - \ln(\tan\frac{\theta}{2}) + \lambda s \cos\theta \end{aligned} \tag{3.5}$$

where λ corresponds to the magnetic mass. The variables θ, ϕ and ψ are angular ones in the polar coordinates with the range $0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi$ and $0 \leq \psi \leq 4\pi$. The s is the square of the radial variable. The corresponding Kähler metric in the coordinate (3.5) is given by

$$g_{1\bar{1}} = \frac{s}{4}\Big(\frac{\cos^2\theta}{1+\lambda s} + (1+\lambda s)\sin^2\theta\Big), \quad g_{1\bar{2}} = \frac{s}{4}\frac{\cos\theta}{1+\lambda s}, \quad g_{2\bar{2}} = \frac{s}{4}\frac{1}{1+\lambda s}. \tag{3.6}$$

These clarify the relation of the eqn.(3.3) to the eqn.(3.4) as

$$\tilde{K} = \frac{s}{2} = K^{TN} \mid_{\lambda=0} \ .$$

In this way we see that the quasi-Kähler potential (3.2) which is solved via the continual Toda equation is dual to the hyper-Kähler Taub-NUT metric with zero magnetic mass. From this fact a question arises whether there also exist the relation of the continual Toda equation to the hyper-Kähler Taub-NUT metric with $\lambda \neq 0$. We show below that this is the case.

Since we have a hyper-Kähler Taub-NUT metric associated with the Kähler potential (3.4) in the coordinate (3.5) which is dual to a solution of (2.9) in the case of $\lambda = 0$, we try to dualize (3.4) with respect to $U(1)_{z^1}$ isometry. Due to the Z_2 property of duality transformation, we can construct the Kähler potential (3.4) from the resulting quasi-Kähler potential.

Here we deal with the case that the Kähler geometry which is described by two chiral superfields is dualized to the quasi-Kähler geometry described by a chiral and a twisted chiral superfield. This causes the modification of the duality transformation (B.2) and (B.3) as follows. The dual quasi-Kähler potential, interchanging a chiral superfield Z^1 with a twisted chiral superfield W, is determined by

$$K = K^{TN} + (z^1 + \bar{z}^1)(w + \bar{w}),$$

with a constraint $0 = \partial_{z^1} K^{TN} + (w + \bar{w})$. The constraint becomes $(w + \bar{w}) = -\frac{1}{4}s(1 + \frac{1}{2}\lambda s \sin^2\theta)$ and we obtain the quasi-Kähler potential

$$K = \frac{s}{2}[1 + \frac{1}{4}\lambda s(1 + \cos^2\theta) - (1 + \frac{1}{2}\lambda s \sin^2\theta)\ln(s\sin\theta)], \tag{3.7}$$

where the quasi-Kähler coordinate is given, denoting z^2 as $8z$, by

$$\begin{aligned} z &= \frac{1}{8}\big(i\psi - \ln(\tan\frac{\theta}{2}) + \lambda s\cos\theta\big), \\ w &= \frac{i}{2}\phi - \frac{1}{8}s(1 + \frac{1}{2}\lambda s\sin^2\theta). \end{aligned} \tag{3.8}$$

In order to show that the quasi-Kähler potential (3.7) arises as the solution of the continual Toda equation, we compute the derivatives of K to be given by

$$\begin{aligned} K_z &= 2s\cos\theta, \\ K_w &= 2\ln(s\sin\theta), \\ K_{z\bar{z}} &= 8s\sin^2\theta\Big(\frac{\cos^2\theta}{1+\lambda s} + (1+\lambda s)\sin^2\theta\Big)^{-1}, \\ K_{w\bar{w}} &= -\frac{8}{s}\Big(\frac{\cos^2\theta}{1+\lambda s} + (1+\lambda s)\sin^2\theta\Big)^{-1}, \\ K_{z\bar{w}} &= -8\frac{\cos\theta}{1+\lambda s}\Big(\frac{\cos^2\theta}{1+\lambda s} + (1+\lambda s)\sin^2\theta\Big)^{-1}. \end{aligned} \tag{3.9}$$

One can easily see that eqns.(2.9) and (2.11) are satisfied. Thus the potential (3.7) describes $N = 2$ superstring background. The corresponding geometry which is associated with the potential (3.7) is given by

$$
\begin{aligned}
ds^2 &= \frac{16}{s}\Big(\frac{\cos^2\theta}{1+\lambda s} + (1+\lambda s)\sin^2\theta\Big)^{-1}\big(dwd\bar{w} + s^2\sin^2\theta dz d\bar{z}\big),\\
H_{z\bar{z}w} &= -H_{z\bar{z}\bar{w}} = +32\sin^2\theta\Big(\frac{\cos^2\theta}{1+\lambda s} + (1+\lambda s)\sin^2\theta\Big)^{-3}\\
&\qquad\qquad \times\Big(\sin^2\theta + \frac{\cos^2\theta}{1+\lambda s} + \frac{2\lambda s}{(1+\lambda s)^2} - \frac{\lambda^2 s^2\sin^2\theta}{(1+\lambda s)^3}\Big),\\
H_{w\bar{w}z} &= -H_{w\bar{w}\bar{z}} = -32\lambda\sin^2\theta\cos\theta\Big(\frac{\cos^2\theta}{1+\lambda s} + (1+\lambda s)\sin^2\theta\Big)^{-3}\\
&\qquad\qquad \times\Big(\frac{3}{1+\lambda s} - \frac{\lambda^2 s^2\cos^2\theta}{(1+\lambda s)^3}\Big),\\
2\Phi &= -\ln s - \ln\Big|\frac{\cos^2\theta}{1+\lambda s} + (1+\lambda s)\sin^2\theta\Big| + \text{const.}
\end{aligned}
\tag{3.10}
$$

In the real coordinate the metric is expressed as

$$
\begin{aligned}
ds^2 =& \frac{1+\lambda s}{4s}(ds^2 + s^2 d\theta^2)\\
&+ \frac{1}{4s}\Big(\frac{\cos^2\theta}{1+\lambda s} + (1+\lambda s)\sin^2\theta\Big)^{-1}(16d\phi^2 + s^2\sin^2\theta d\psi^2),
\end{aligned}
$$

where ϕ and ψ are introduced in (3.8) to be the imaginary part of w and z respectively. One find that in the case $\lambda = 0$ the metric is dual with respect to rotational $U(1)$ isometry associated with the angular coordinate ϕ to flat metric as expected. The corresponding scalar curvature is

$$
\begin{aligned}
R =& (24 + 176\lambda s + 312\lambda^2 s^2 + 52\lambda^3 s^3 - 235\lambda^4 s^4 - 168\lambda^5 s^5 - 32\lambda^6 s^6\\
&+ 144\lambda s\cos 2\theta + 488\lambda^2 s^2\cos 2\theta + 592\lambda^3 s^3\cos 2\theta + 388\lambda^4 s^4\cos 2\theta\\
&+ 160\lambda^5 s^5\cos 2\theta + 32\lambda^6 s^6\cos 2\theta - 4\lambda^3 s^3\cos 4\theta + 7\lambda^4 s^4\cos 4\theta\\
&+ 8\lambda^5 s^5\cos 4\theta)\Big/\Big(4s(1+\lambda s)^5\Big(\frac{\cos^2\theta}{1+\lambda s} + \sin^2\theta(1+\lambda s)\Big)^2\Big).
\end{aligned}
$$

It follows that the scalar curvature is asymptotically zero ($s \to +\infty$). For the case $\lambda = 0$, the curvature singularity is at $s = 0$. In turn if we consider $\lambda < 0$, there are two singularities at $s = 0$ and $-1/\lambda$.

We obtained the non-trivial superstring background which is a solution of eqn.(2.9) and is dual to the hyper-Kähler Taub-NUT solution. As is mentioned in the section 2 the eqn.(2.9) is related to the continual Toda equation, so that a class of the solution of the continual Toda equation emerge. The corresponding solution of the continual Toda equation (2.11) : $\partial_z\partial_{\bar{z}}\mathcal{U} + \partial_w^2 e^{\mathcal{U}} = 0$ is

$$\mathcal{U} = 2\ln(s\sin\theta), \tag{3.11}$$

where the r.h.s. is expressed implicitly in terms of z and w by means of eqn.(3.8) . If the term $\partial_z\partial_{\bar{z}}\mathcal{U}$ is independent of the variable w: $\partial_w(\partial_z\partial_{\bar{z}}\mathcal{U}) = 0$, the continual Toda equation can be reduced to the Liouville equation. We find that the solution (3.11) is not the case and can not be reduced to the one of the Liouville equation.

4. The Quasi-Kähler Geometry and The Real Heavens

In section 2 and 3 we present $N = 2$ superstring quasi-Kähler backgrounds which are dual to Eguchi-Hanson instanton and Taub-NUT instanton respectively. Both are obtained through solutions of the continual Toda equation. In this section we study the origin of the integrable property of the quasi-Kähler geometry for the case(i) and case(ii). It is shown that the integrability can be understand as a direct reflection of the one of the real heavens: real, self-dual, euclidean, Einstein spaces. It was shown, in ref.[13], that all solutions to the real vacuum Einstein equations with self-dual or anti-self-dual curvature:

$$R_{\mu\nu\rho\sigma} = \pm\frac{1}{2}\sqrt{\det G}\ \epsilon_{\rho\sigma}{}^{\kappa\lambda}R_{\mu\nu\kappa\lambda},$$

in the presence of at least one Killing symmetry, fall into two cases which correspond to two distinct types of Killing vectors. The first type, what is called "translational", corresponds to Killing vectors K_ν with self-dual or anti-self-dual covariant derivatives

$$\nabla_\mu K_\nu = \pm\frac{1}{2}\sqrt{\det G}\ \epsilon_{\mu\nu}{}^{\kappa\lambda}\nabla_\kappa K_\lambda.$$

The second type, what is called "rotational", includes all other Killing vectors.

These gravitational backgrounds are hyper-Kähler and consistent with $N = 4$ world sheet supersymmetry. The relation of the world sheet supersymmetry to T-duality transformation is recently considered in ref.[17] by using these backgrounds.

In the following we show that the quasi-Kähler backgrounds for the case(i) and case(ii) are dual to the real heavens with (at least) one "translational" Killing symmetry and "rotational" one respectively.

At first we consider the case(i). The string backgrounds admit a conformally flat metric coupled to axionic instanton and have been considered before[5,15,16]. To make the statement concrete we recall the quasi-Kähler backgrounds:

$$
\begin{aligned}
& ds^2 = 2K_{u\bar{u}}dud\bar{u} - 2K_{v\bar{v}}dvd\bar{v} = 2K_{u\bar{u}}(dud\bar{u} + dvd\bar{v}), \\
& H_{u\bar{u}v} = -\partial_u\partial_{\bar{u}}\partial_v K, \qquad H_{u\bar{u}\bar{v}} = +\partial_u\partial_{\bar{u}}\partial_{\bar{v}} K, \\
& H_{v\bar{v}u} = +\partial_v\partial_{\bar{v}}\partial_u K, \qquad H_{v\bar{v}\bar{u}} = -\partial_v\partial_{\bar{v}}\partial_{\bar{u}} K, \\
& 2\Phi = \ln K_{u\bar{u}} + \text{const.} = \ln(-K_{v\bar{v}}) + \text{const.}
\end{aligned}
\tag{4.1}
$$

with K satisfying the flat Laplace equation $(\partial_u\partial_{\bar{u}} + \partial_v\partial_{\bar{v}})K = 0$. If there exist $U(1)_v$ isometry ($U(1)_u$ isometry), denoting the Killing vector as $\partial/\partial\tau$, the real coordinates (τ, x, y, z) are introduced by

$$
v = z + i\tau, \quad u = x + iy \qquad (u = z + i\tau, \quad v = x + iy).
$$

In the following we distinguish these cases by means of upper (lower) sign. In these coordinates eqns.(4.1) are expressed by the following

$$
\begin{aligned}
& ds^2 = g_{\tau\tau}(d\tau + A_i dx^i)^2 + \bar{g}_{ij}dx^i dx^j, \\
& g_{\tau\tau} = 2K_{u\bar{u}}, \quad \bar{g}_{ij} = 2K_{u\bar{u}}\delta_{ij}, \quad A_i = 0, \\
& H_{\tau xy} = \mp 2\partial_z K_{u\bar{u}}, \quad H_{\tau yz} = \mp 2\partial_x K_{u\bar{u}}, \quad H_{\tau zx} = \mp 2\partial_y K_{u\bar{u}}, \\
& 2\Phi = \ln K_{u\bar{u}} + \text{const.}
\end{aligned}
\tag{4.2}
$$

Denoting $2K_{u\bar{u}}$ as V, the anti-symmetric tensor $B_{\mu\nu}$ can be chosen as $B_{\tau i} = \omega_i$ with satisfying the special condition: $\nabla V = \pm\nabla \times \omega$.

Now we perform the duality transformation with respect to τ direction:

$$\begin{aligned}
&\tilde{g}_{\tau\tau} = 1/g_{\tau\tau}, \quad \tilde{A}_i = B_{\tau i}, \quad \tilde{B}_{\tau i} = A_i, \\
&\tilde{B}_{ij} = B_{ij} - 2A_{[i}B_{j]\tau}, \\
&2\tilde{\Phi} = 2\Phi - \ln g_{\tau\tau}, \quad \tilde{\bar{g}}_{ij} = \bar{g}_{ij}.
\end{aligned} \tag{4.3}$$

The resulting dual backgrounds are given by

$$d\tilde{s}^2 = \frac{1}{V}(d\tau + \omega_i dx^i)^2 + V(dx^2 + dy^2 + dz^2) \tag{4.4}$$

where ω_i are constrained to satisfy the condition

$$\nabla V = \pm \nabla \times \omega, \tag{4.5}$$

hence V satisfys the flat Laplace equation. It was shown in ref.[13] that, in the presence of (at least) one "translational" Killing symmetry, solutions to the real vacuum Einstein equations with self-dual or anti-self-dual curvature are completely determined by V satisfying the condition (4.5) with metric (4.4) [14]. Localized solutions of the flat Laplace equation correspond to the multi-asymptotically locally euclidean instantons or the multi-Taub-NUT instantons, depending on the asymptotic nature.

We next consider the case(ii). The quasi-Kähler backgrounds have the following form, denoting $K_w = \partial_{w+\bar{w}} K \equiv \mathcal{U}(u, \bar{u}, w + \bar{w})$,

$$\begin{aligned}
&ds^2 = -2\partial_{w+\bar{w}}\, \mathcal{U} dw d\bar{w} - 2\partial_{w+\bar{w}}\, \mathcal{U} e^{\mathcal{U}} du d\bar{u}, \\
&H_{u\bar{u}w} = -H_{u\bar{u}\bar{w}} = -\partial_u \partial_{\bar{u}} \mathcal{U}, \quad H_{w\bar{w}u} = \partial_{w+\bar{w}} \partial_u \mathcal{U}, \quad H_{w\bar{w}\bar{u}} = -\partial_{w+\bar{w}} \partial_{\bar{u}} \mathcal{U}, \\
&2\Phi = \ln \partial_{w+\bar{w}} \mathcal{U},
\end{aligned} \tag{4.6}$$

with $\mathcal{U}$ satisfying the continual Toda equation $\partial_u \partial_{\bar{u}} \mathcal{U} + \partial^2_{w+\bar{w}}\, e^{\mathcal{U}} = 0$. It is convenient to introduce the real coordinates (τ, x, y, z) as

$$w = -(z + i\tau), \quad u = \begin{cases} y + ix, \\ x + iy. \end{cases} \tag{4.7}$$

We denote the upper and lower case to correspond to the upper and lower sign in

the following. In these coordinates eqns.(4.6) are expressed by

$$
\begin{aligned}
&ds^2 = g_{\tau\tau}(d\tau + A_i dx^i)^2 + \bar{g}_{ij}dx^i dx^j, \\
&g_{\tau\tau} = \partial_z \mathcal{U}, \quad A_i = 0, \\
&\bar{g}_{ij} = \mathrm{diag}(\partial_z \mathcal{U} e^{\mathcal{U}}, \partial_z \mathcal{U} e^{\mathcal{U}}, \partial_z \mathcal{U}), \\
&H_{\tau xy} = \mp(\partial_x^2 + \partial_y^2)\mathcal{U}, \quad H_{\tau xz} = \mp\partial_z\partial_y \mathcal{U}, \quad H_{\tau yz} = \pm\partial_z\partial_x \mathcal{U}, \\
&2\Phi = \ln \partial_z \mathcal{U} + \mathrm{const.}
\end{aligned}
\tag{4.8}
$$

with $\mathcal{U}$ satisfying $(\partial_x^2 + \partial_y^2)\mathcal{U} + \partial_z^2 \ e^{\mathcal{U}} = 0$. The torsion fields are compatible with choosing the anti-symmetric tensor $B_{\mu\nu}$ as

$$
B_{\tau x} = \mp\partial_y \mathcal{U}, \quad B_{\tau y} = \pm\partial_x \mathcal{U},
$$

and the other components are zero.

Now the Killing vector corresponding to the $U(1)_w$ isometry is $\partial/\partial\tau$ and we perform the duality transformation (4.3) . The resulting dual backgrounds are given by

$$
d\tilde{s}^2 = \frac{1}{\partial_z \mathcal{U}}(d\tau \mp \partial_y \mathcal{U} dx \pm \partial_x \mathcal{U} dy)^2 + \partial_z \mathcal{U}[e^{\mathcal{U}}(dx^2 + dy^2) + dz^2] \tag{4.9}
$$

with $\mathcal{U}$ satisfying the continual Toda equation

$$
(\partial_x^2 + \partial_y^2)\mathcal{U} + \partial_z^2 \ e^{\mathcal{U}} = 0. \tag{4.10}
$$

It was shown in ref.[13] that, in the presence of (at least) one “rotational” Killing symmetry, solutions to the real vacuum Einstein equations with self-dual or anti-self-dual curvature are completely determined by $\mathcal{U}$ satisfying eqn.(4.10) with the metric (4.9) . In the above, we considered the case that the quasi-Kähler backgrounds have one $U(1)$ isometry with respect to a twisted chiral superfield. If we consider the case $C_1 = 0, C_2 \neq 0$ instead of the case(ii) $C_1 \neq 0, C_2 = 0$ in section

2, the quasi-Kähler backgrounds turn to possess one $U(1)$ isometry with respect to a chiral superfield. In this case, denoting w and u as the lowest component of a chiral and twisted chiral superfield respectively, the metric and torsion fields $H_{\mu\nu\rho}$ have opposite sign to eqns.(4.6) . Introducing the real coordinates (4.7) with the change $w \to -w$, the dual backgrounds have the metric (4.9) with the constraint (4.10) again.

As a consequence, we can state that the origin of the integrability of the quasi-Kählerian for the case(i) and case(ii) lies in the real heavens.

In section 2 and 3, the quasi-Kähler backgrounds which are dual to the Eguchi-Hanson and Taub-NUT instanton background respectively are constructed. Since these instanton backgrounds admit not only "translational" Killing symmetry but also "rotational" one, they can be written in the form (4.9) . The multi-ALE and multi-Taub-NUT instanton backgrounds don't admit additional "rotational" Killing symmetry in general except for the Eguchi-Hanson and Taub-NUT instanton backgrounds. Hence the quasi-Kähler backgrounds which are dual to these instantons can not be constructed for the case(ii).

5. Summary and Discussions

In this section, we first summarize our result and then briefly discuss their generalizations.

We investigate four dimensional $N = 2$ superstring backgrounds which are described by a chiral superfield and a twisted chiral one. In particular we considered the case where there is (at least) one Killing symmetry and the quasi-Kähler potential is determined by the continual Toda equation. We found that the background which is dual to the well-known Taub-NUT instanton background arises through a non-trivial solution to the continual Toda equation. We clarify the relationship of the quasi-Kähler backgrounds with the real heavens i.e. the real, self-dual, euclidean, Einstein spaces. It is found that the quasi-Kähler backgrounds for the

case(i) and (ii) are dual to the real heavens with a "translational" Killing symmetry and a "rotational" one respectively. Then it was found that the origin of the integrable property lies in the real heavens.

Since the hyper-Kähler Taub-NUT and Eguchi-Hanson instanton background is known to be consistent with $N = 4$ world sheet supersymmetry, we may expect that the corresponding quasi-Kähler backgrounds are exact to all orders of α' if the duality transformation preserve $N = 4$ world sheet supersymmetry. In ref.[17, 18] the relation of world sheet supersymmetry and T-duality transformation is considered.

Four-dimensional $N = 2$ superstring backgrounds described by the (2,2) σ-models formulated in terms of a chiral and a twisted chiral superfields include another case than the case(i) and (ii) studied in this paper. For the case(iii), where there are at least two $U(1)$ Killing symmetry and the quasi-Kähler potential is determined by a non-linear differential equation, we find several interesting backgrounds. One of them is a class of conformally flat backgrounds with (2,2) signature which have non-trivial dilaton and torsion fields. The dual backgrounds are hyper-Kähler nevertheless possesses linear dilaton field. Another interesting class is the direct products of two 2-dim backgrounds. The product of $SU(2)/U(1) \otimes SL(2, R)/U(1)$ which is known to possess $N = 4$ world sheet supersymmetry falls into this class. The case(iii) covers these non-trivial solutions but the general solution is unknown.

We studied (2,2) σ-models described in terms of twisted chiral and chiral superfields. It is known that these σ-models put a strong restriction on the background geometry, namely, two complex structures must *commute*. The two *commuting* complex structures emerge when one consider the WZNW σ models only on $SU(2) \otimes U(1)$ or $U(1)^4$ among the various group manifolds[11]. Thus the generic (2,2) supersymmetric σ-models can not be exhausted employing these superfields.

In ref.[12] the (2,2) σ-models formulated in terms of semi-chiral superfields,

which satisfy only a left-handed or right-handed chirality condition but not both simultaneously, are shown to possess two non-commuting complex structures and correspond to the generic case. So far only the case of (2,2) world sheet supersymmetry has been considered. If heterotic (2,0) σ-models are considered, the metric and torsion are given by a complex vector potential[10]. The two dimensional action is formulated in terms of (left-handed or right-handed) chiral superfields in which the vector potential appears.

It is intriguing problem for us to investigate the backgrounds which are described by these σ-models in the string context.

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APPENDIX.A

In this section we consider the vanishing conditions of β-functions imposed on K. The equations to be solved are ten for $\beta^G_{\mu\nu}$, six for $\beta^B_{\mu\nu}$ and one for β^Φ. These were re-expressed, defining $\boldsymbol{U} = \log K_{u\bar{u}}$ and $\boldsymbol{V} = \log K_{v\bar{v}}$, in terms of $\boldsymbol{U}, \boldsymbol{V}$ and Φ entirely in ref.[5].

It follows from the six vanishing conditions of β^G_{uu} , β^G_{uv} , $\beta^G_{u\bar{v}}$, β^G_{vv} , β^B_{uv} and $\beta^B_{u\bar{v}}$ that

$$\begin{aligned} \partial_u \boldsymbol{V} &= 2\partial_u \Phi + \bar{C}_1(\bar{u}) \exp \boldsymbol{U}, \\ \partial_v \boldsymbol{U} &= 2\partial_v \Phi + \bar{C}_2(\bar{v}) \exp \boldsymbol{V}, \end{aligned} \tag{A.1}$$

where $\bar{C}_1(\bar{u})$ and $\bar{C}_2(\bar{v})$ are arbitrary anti-holomorphic functions. The complex conjugate of the above six vanishing conditions tell us that analogous equations with $C_1(u)$ and $C_2(v)$ hold for the derivatives with respect to $\bar{u}$ and $\bar{v}$.

The remaining five conditions are the vanishing of $\beta^G_{u\bar{u}}$, $\beta^G_{v\bar{v}}$, $\beta^B_{u\bar{u}}$, $\beta^B_{v\bar{v}}$ and β^Φ. In order to proceed with taking into account of these conditions,the following

three exclusive cases were considered: (i) $C_1 = C_2 = 0$, (ii) $C_1 = 0$, $C_2 \neq 0$, (iii) $C_1, C_2 \neq 0$.

In the following we concentrate ourselves to the case(i) and case(ii).

For the case(i), the conditions (2.5) are re-expressed by the following set of differential equations:

$$\begin{aligned} &\partial_u(\boldsymbol{V} - 2\Phi) = \partial_{\bar{u}}(\boldsymbol{V} - 2\Phi) = 0, \quad \partial_v\partial_{\bar{v}}(\boldsymbol{V} - 2\Phi) = 0, \\ &\partial_v(\boldsymbol{U} - 2\Phi) = \partial_{\bar{v}}(\boldsymbol{U} - 2\Phi) = 0, \quad \partial_u\partial_{\bar{u}}(\boldsymbol{U} - 2\Phi) = 0, \end{aligned} \qquad (A.2)$$

which can be solved by $\boldsymbol{U} - 2\Phi =$ const. and $\boldsymbol{V} - 2\Phi =$ const. The potential K must satisfy

$$K_{u\bar{u}} = K_{v\bar{v}} e^c \qquad (A.3)$$

where c is a constant. The dilaton field is expressed as

$$2\Phi = \ln |K_{u\bar{u}}| + \text{const.} \qquad (A.4)$$

Without any loss of generality, we can consider the constant c is pure imaginary since the real part of it can be absorbed by rescaling v. Moreover, it is restricted to 0 or $i\pi$ for the nontrivial solution. For the case $c = 0$ the corresponding backgrounds have (2,2) signature. To obtain euclidean backgrounds we must choose $c = i\pi$. In this case, the eqn.(A.3) is nothing but the Laplace equation.

For the case(ii), it is very useful to perform the following change of coordinates [5]:

$$w = \int \frac{\mathrm{d}v}{C_2(v)}. \qquad (A.5)$$

Combining a remaining condition of $\beta^B_{u\bar{u}} = 0$ with eqns.(A.1) we obtain that $\Phi = \Phi(u, \bar{u}, w + \bar{w})$, $\boldsymbol{V} = \boldsymbol{V}(u, \bar{u}, w + \bar{w})$ and $\boldsymbol{U} = \boldsymbol{U}(u, \bar{u}, w + \bar{w})$. Thus this case

necessarily leads to at least one $U(1)$ isometry. The sixteen conditions for (2.5) are entirely re-expressed by the following set of differential equations:

$$\partial_u(\tilde{\boldsymbol{V}} - 2\Phi) = \partial_{\bar{u}}(\tilde{\boldsymbol{V}} - 2\Phi) = 0, \tag{A.6}$$

$$\partial_w(\boldsymbol{U} - 2\Phi) = \exp \tilde{\boldsymbol{V}}, \tag{A.7}$$

$$\partial_w^2(\tilde{\boldsymbol{V}} - 2\Phi) = 0, \tag{A.8}$$

$$\partial_u\partial_{\bar{u}}(\boldsymbol{U} - 2\Phi) = \partial_w \exp \boldsymbol{U}, \tag{A.9}$$

where we denote $\tilde{\boldsymbol{V}} = \ln K_{ww} = \ln(C_2(v)\bar{C}_2(v)K_{v\bar{v}})$. The last two equations follow from the vanishing of $\beta^G_{u\bar{u}}$ and $\beta^G_{v\bar{v}}$. The condition $\beta^B_{v\bar{v}} = 0$ leads no additional condition. Eqns.(A.6) and (A.8) can be solved to give

$$\tilde{\boldsymbol{V}} - 2\Phi = c_1(w + \bar{w}) + c_2, \tag{A.10}$$

where c_1 and c_2 are integration constants. Integrating (A.7) with respect to w and (A.9) with respect to u and $\bar{u}$ lead to the equation: $\ln K_{u\bar{u}} = 2\Phi + K_w + \lambda(u) + \bar{\lambda}(\bar{u})$ where $\lambda(u)$ and $\bar{\lambda}(\bar{u})$ arise as integration "constants". Eliminating Φ by using eqn.(A.10), the potential K has to satisfy

$$K_{ww} = K_{u\bar{u}} e^{-K_w + c_1(w+\bar{w}) + c_2 - \lambda(u) - \bar{\lambda}(\bar{u})}. \tag{A.11}$$

The terms $\lambda(u)$ and $\bar{\lambda}(\bar{u})$ can be absorbed by the holomorphic coordinate transformation

$$e^{\lambda(u)} du = dF, \quad e^{\bar{\lambda}(\bar{u})} d\bar{u} = d\bar{F}, \tag{A.12}$$

which transform the eqn.(A.11) to

$$K_{ww} = K_{u\bar{u}} e^{-K_w + c_1(w+\bar{w}) + c_2}, \tag{A.13}$$

where we denote $(F, \bar{F})$ as $(u, \bar{u})$ again. Eqn.(A.13) determines target space geometry. The dilaton field is given in terms of a solution of eqn.(A.13) to be

$$2\Phi = \ln K_{w\bar{w}} - c_1(w + \bar{w}) - c_2. \tag{A.14}$$

Using the above expressions (A.13) and (A.14) one can find that the central charge

deficit δc is proportional to the constant c_1: $\delta c = -3\alpha' c_1$ and thus we set $c_1 = 0$ from now on. As a consequence, in order that one-loop β-functions (2.5) and (2.6) vanish, for case(ii) it is the differential equation (A.13) with $c_1 = 0$ that determine the string background. The dilaton field Φ is determined by the solution of (A.13) with $c_1 = 0$ through eqn.(A.14).

The (quasi-)Kähler transformation is a gauge transformation for the spacetime geometry, but transformed potential K for the case(ii) is not always a solution of the same equation. Let us spend the rest of the present subsection to comment on the quasi-Kähler transformation. Here we consider the quasi-Kähler transformation in the presence of $U(1)$ isometry with respect to W. In this case the transformation (2.3) is classified by the following three cases:

$$K \to \hat{K} = K - (w + \bar{w})(\lambda(u) + \bar{\lambda}(\bar{u})), \tag{A.15}$$

$$K \to \hat{K} = K - k(w + \bar{w}), \tag{A.16}$$

$$K \to \hat{K} = K - \lambda(u) - \bar{\lambda}(\bar{u}). \tag{A.17}$$

Let K be a solution of (A.13). The potential $\hat{K}$ which is generated from K under the quasi-Kähler transformation (A.15) satisfys not eqn.(A.13) but (A.11). As is mentioned before, eqn.(A.11) is transformed to the original equation under the coordinate transformation (A.12). Thus the transformed K is also a solution only when the quasi-Kähler transformation (A.15) is followed by the coordinate transformation (A.12). The Kähler transformation (A.16) make the constant c_2 to be shifted by a real constant $-k$. To obtain the solution of original equation we perform the coordinate transformation

$$e^{k/2} du = dF, \quad e^{k/2} d\bar{u} = d\bar{F}. \tag{A.18}$$

In turn the Kähler transformation (A.17) is invariance of (A.13).

Since a real part of c_2 can be absorbed by the quasi-Kähler transformation (A.16) or the coordinate transformation (A.18), the imaginary part of c_2 is relevant.

If we consider the case $c_2 = 0$, the resulting geometry has $(2,2)$ signature. In order to have euclidean signature we must set $c_2 = i\pi$.

APPENDIX.B

In this section we consider duality transformation. As was explained in ref.[3,4] this duality can be described by interchanging twisted chiral superfields with chiral ones. Let us consider the case that the potential K has one Killing symmetry with respect to V and is of the form

$$K = K(U, \bar{U}, V + \bar{V}), \tag{B.1}$$

where V is a twisted chiral field, whereas U is a chiral field. We denote the above $U(1)$ isometry as $U(1)_v$ for simplicity.

The 'dual' potential $\tilde{K}$ is obtained as a Legendre transform of K;

$$\tilde{K}(U, \bar{U}, V + \bar{V}, \Psi + \bar{\Psi}) = K(U, \bar{U}, V + \bar{V}) - (V + \bar{V})(\Psi + \bar{\Psi}), \tag{B.2}$$

with

$$\frac{\partial K}{\partial v} - (\psi + \bar{\psi}) = 0, \tag{B.3}$$

where Ψ is a chiral field. Since the dual potential $\tilde{K}$ is described by two chiral superfields, the dual transformation explained above produces a torsionless Kähler manifold. It follows that the dual metric has the following form:

$$\tilde{G}_{\mu\nu} = \begin{pmatrix} 0 & \tilde{K}_{\psi\bar{\psi}} & 0 & \tilde{K}_{\psi\bar{v}} \\ \tilde{K}_{\psi\bar{\psi}} & 0 & \tilde{K}_{v\bar{\psi}} & 0 \\ 0 & \tilde{K}_{v\bar{\psi}} & 0 & \tilde{K}_{v\bar{v}} \\ \tilde{K}_{\psi\bar{v}} & 0 & \tilde{K}_{v\bar{v}} & 0 \end{pmatrix}. \tag{B.4}$$

On the other hand if the Killing symmetry is with respect to a chiral superfield U, the corresponding dual metric has opposite signature to (B.4).

Since we are considering the case that there are one chiral and one twisted chiral superfield, the duality transformation produces a torsionless Kählar manifold explained above. In order that this N=2 preserving duality transformation by means of a Legendre transformation coincides with the usual abelian T-duality transformation[7], the dual dilaton field must be

$$2\tilde{\Phi} = 2\Phi - \ln 2K_{vv}. \qquad (B.5)$$

Since the dilaton field is expressed by eqn.(2.10) for the case(ii), we obtain linear dilaton backgrounds after the duality transformation with respect to $U(1)_w$ isometry. For the case(i), if there exist a $U(1)$ Killing symmetry, the dual dilaton field is constant.

REFERENCES

1. B. Zumino, *Phys. Lett.***B87** (1979) 203.
2. Ph. Spindel, A. Sevrin, W. Troost and A. Van Proeyen, *Nucl. Phys.***B308** (1988) 662.
3. U. Lindström and M. Roček, *Nucl. Phys.***B222** (1983) 285.
4. S. Gates, C. Hull and M. Roček, *Nucl. Phys.***B248** (1984) 157.
5. E. Kiritsis, C. Kounnas and D. Lüst, *Int. J. Mod. Phys.***A9** (1994) 1361.
6. G.L. Cardoso and D. Lüst , *Phys. Lett.***B345** (1995) 220.
7. T. Buscher, *Phys. Lett.***B201** (1988) 466.
8. E. Fradkin and A. Tseytlin, *Nucl. Phys.***B261** (1985) 1; C. Callan, D. Friedan, E. Martinec and M. Perry, *Nucl. Phys.***B262** (1985) 593.
9. G. Bonneau and G. Valent, *Class. Quant. Grav.* **11** (1994) 1133.
10. C. Hull and E. Witten, *Phys. Lett.***B160** (1985) 398.
11. M. Roček, K. Schoutens and A. Sevrin, *Phys. Lett.***B265** (1991) 303.
12. T. Buscher, U. Lindström and M. Roček, *Phys. Lett.***B202** (1988) 94.
13. C. Boyer and J. Finley, *J. Math. Phys.* **23** (1982) 1126; J. Gegenberg and A. Das, *Gen. Rel. Grav.* **16** (1984) 817.
14. S. Hawking, *Phys. Lett.***A60** (1977) 81; G. Gibbons and S. Hawking, *Comm. Math. Phys.***66** (1979) 291; K. Tod and R. Ward, *Proc. R. Soc. London Ser.* **A368** (1979) 411.
15. A. Dabholkar, G. Gibbons, J. A. Harvey and F. Ruiz-Ruiz, *Nucl. Phys.***B340** (1990) 33; A. Strominger, *Nucl. Phys.***B343** (1990) 167; C. Callan, J. A. Harvey and A. Strominger, *Nucl. Phys.***B359** (1991) 611, **B367** (1991) 60.
16. S.J. Rey, *Phys. Rev.***D43** (1991) 526; M. Duff and J. X. Lu, *Nucl. Phys.***B354** (1991) 141; R. Khuri, *Nucl. Phys.***B387** (1992) 315.

17. I. Bakas, *Phys. Lett.***B343** (1995) 103; I. Bakas and K. Sfetsos, *Phys. Lett.***B349** (1995) 448

18. S. F. Hassan, CERN-TH-95-98, hep-th/9504148.